Method 1

Tim "The Golden Boy" Noo

The first square we cut out has as its sidelength the shorter dimension of the original rectangle. Hence S_2 (the second square we cut out) has side length $\phi - 1$. But since ϕ satisfies the the identity $x^2 - x - 1 = 0$, we know that $\phi - 1 = \frac{1}{\phi}$, so $S_2 = \frac{1}{\phi} = \frac{1}{\phi^{2-1}}$.

Indeed S_n (the sidelength of the n^{th} cut out square) has the property that $S_n = \frac{1}{\phi^{n-1}}$. This is due to the fact that $\frac{S_n}{S_{n-1}} = \phi$, and a proof by induction, while easy, is omitted here.

Now, we notice that since this survivor exists we can find it. Note that every 4^{th} square, starting with the first, increases our lower bound of where the x-coordinate of the survivor lives. Thus the limit of the sum of the side lengths of these squares will give us the x-coordinate of the survivor.

Now we need the limit of $1 + \frac{1}{\phi^4} + \frac{1}{\phi^8} + \frac{1}{\phi^{12}} + \ldots$ which can be symbolized by $\lim_{k \to \infty} \sum_{i=1}^{k} \frac{1}{\phi^{4(i-1)}}$. Note that this is a geometric series with ratio $r = \frac{1}{\phi^4}$, & $0 < \frac{1}{\phi^4} < 1$ so the sum converges. First term is 1

$$\text{so } \lim_{k \to \infty} \sum_{i=1}^{k} \frac{1}{\phi^{4(i-1)}} = \sum_{i=1}^{\infty} \frac{1}{\phi^{4(i-1)}} = \frac{1}{1 - 1/\phi^4} = \frac{1}{\frac{\phi^4 - 1}{\phi^4}} = \frac{\phi^4}{\phi^4 - 1}.$$

But there is a nice property between ϕ & the Fibonacci numbers, namely that $\phi^n = F_n \phi + F_{n-1}$. So in the above sum we obtain $\frac{\phi^4}{\phi^4 - 1} = \frac{3\phi + 2}{3\phi + 1}$. Now, $\phi = \frac{1+\sqrt{5}}{2}$ so this becomes $\frac{3\left(\frac{1+\sqrt{5}}{2}\right) + 2}{3\left(\frac{1+\sqrt{5}}{2}\right) + 1}$

$$\text{Hence, } \frac{\phi^4}{\phi^4 - 1} = \frac{7 + 3\sqrt{5}}{5 + 3\sqrt{5}} \left(\frac{5 - 3\sqrt{5}}{5 - 3\sqrt{5}}\right) = \frac{-10 - 6\sqrt{5}}{-20} = \frac{1}{2} + \frac{3\sqrt{5}}{10}$$

So the x-coordinate of the Golden Survivor is $\frac{1}{2} + \frac{3\sqrt{5}}{10}$.

method 1 cont

Now, we note that every fourth square starting with square 4 increases our lower bound for the y-coordinate of the Golden Survivor. Thus we proceed as before, that is find $\frac{1}{\phi^3}+\frac{1}{\phi^7}+\frac{1}{\phi^{11}}+\dots$

Thus we need $\lim_{k\to\infty}\sum_{i=1}^{k}\frac{1}{\phi^{4i-1}}$. This sum also is geometric with $r=\frac{1}{\phi^4}$

So we get that $\lim_{k\to\infty}\sum_{i=1}^{k}\frac{1}{\phi^{4i-1}} = \sum_{i=1}^{\infty}\frac{1}{\phi^{4i-1}} = \frac{1/\phi^3}{1-1/\phi^4}$ Since the first term is $\frac{1}{\phi^3}$. But $\frac{\frac{1}{\phi^3}}{1-\frac{1}{\phi^4}} = \frac{\phi^4}{\phi^3(\phi^4-1)} = \frac{\phi}{3\phi+1} = \frac{1+\sqrt{5}}{5+3\sqrt{5}}\left(\frac{5-3\sqrt{5}}{5-3\sqrt{5}}\right) = \frac{-10+2\sqrt{5}}{-20}$

So $\sum_{i=1}^{\infty}\frac{1}{\phi^{4i-1}} = \frac{-10+2\sqrt{5}}{-20} = \frac{1}{2}-\frac{\sqrt{5}}{10}$. So the y-coordinate is $\frac{1}{2}-\frac{\sqrt{5}}{10}$.

Hence, the Golden Survivor is the point $\left(\frac{1}{2}+\frac{3\sqrt{5}}{10}, \frac{1}{2}-\frac{\sqrt{5}}{10}\right)$

Note: To find these coordinates, one could evaluate $\phi-\sum_{i=1}^{\infty}\frac{1}{\phi^{4i-2}}$ & $1-\sum_{i=1}^{\infty}\frac{1}{\phi^{4i-3}}$ for x & y coordinates respectively. These just sum up the sidelengths of the squares that lower the upper bounds of the coordinates of the Golden survivor, and subtract them from the width and height of the rectangle respectively.

Method 2

Tim "The Golden Boy" N

Consider the sums we were dealing with before, namely $x = \sum_{k=1}^{\infty} \frac{1}{\phi^{4(k-1)}}$ and $y = \sum_{k=1}^{\infty} \frac{1}{\phi^{4k-1}}$. Let x_n + y_n represent the n^{th} partial sums of x + y respectively, + F_j is the j^{th} Fibonacci number.

Notice $x_1 = 1 = \frac{F_2}{\phi^0} = \frac{F_{2n}}{\phi^{2(n-1)}}$ | $y_1 = \frac{1}{\phi^3} = \frac{F_2}{\phi^{2n+1}} = \frac{F_{2n}}{\phi^{2n+1}}$

$x_2 = \frac{\phi^4+1}{\phi^4} = \frac{3\phi+3}{\phi^4} = \frac{3\phi^2}{\phi^4} = \frac{3}{\phi^2} = \frac{F_{2n}}{\phi^{2(n-1)}}$ | $y_2 = \frac{\phi^4+1}{\phi^7} = \frac{3(\phi^2)}{\phi^7} = \frac{3}{\phi^5} = \frac{F_{2n}}{\phi^{2n+1}}$

$x_3 = \frac{3}{\phi^2} + \frac{1}{\phi^8} = \frac{3\phi^6+1}{\phi^8} = \frac{24\phi+16}{\phi^8} = \frac{8}{\phi^4} = \frac{F_{2n}}{\phi^{2(n-1)}}$ | $y_3 = \frac{3\phi^6+1}{\phi^{11}} = \frac{8(3\phi+2)}{\phi^{11}} = \frac{8\phi^4}{\phi^{11}} = \frac{8}{\phi^7} = \frac{F_{2n}}{\phi^{2n+1}}$

From this you get the idea, that $x_n = \frac{F_{2n}}{\phi^{2(n-1)}}$ + $y_n = \frac{F_{2n}}{\phi^{2n+1}}$.

Thus $\lim_{n\to\infty}(x_n) = \lim_{n\to\infty} \frac{F_{2n}}{\phi^{2n-2}}$ + $\lim_{n\to\infty}(y_n) = \lim_{n\to\infty} \frac{F_{2n}}{\phi^{2n+1}}$

Thus $\lim_{n\to\infty}(x_n) = \phi^2 \lim_{n\to\infty} \frac{F_{2n}}{\phi^{2n}}$ + $\lim_{n\to\infty}(y_n) = \frac{1}{\phi} \lim_{n\to\infty} \frac{F_{2n}}{\phi^{2n}}$

By Binet's Formula $F_j = \frac{1}{\sqrt{5}}\left(\phi^j - \left(-\frac{1}{\phi}\right)^j\right)$

So $\lim_{n\to\infty}(x_n) = \frac{\phi^2}{\sqrt{5}} \lim_{n\to\infty} \frac{\phi^{2n} - \left(-\frac{1}{\phi}\right)^{2n}}{\phi^{2n}}$ + $\lim_{n\to\infty}(y_n) = \frac{1}{\phi\sqrt{5}} \lim_{n\to\infty} \frac{\phi^{2n} - \left(\frac{-1}{\phi}\right)^{2n}}{\phi^{2n}}$

Since $2n$ is even for all n, $\left(-\frac{1}{\phi}\right)^{2n} = \left(\frac{1}{\phi}\right)^{2n}$ Thus

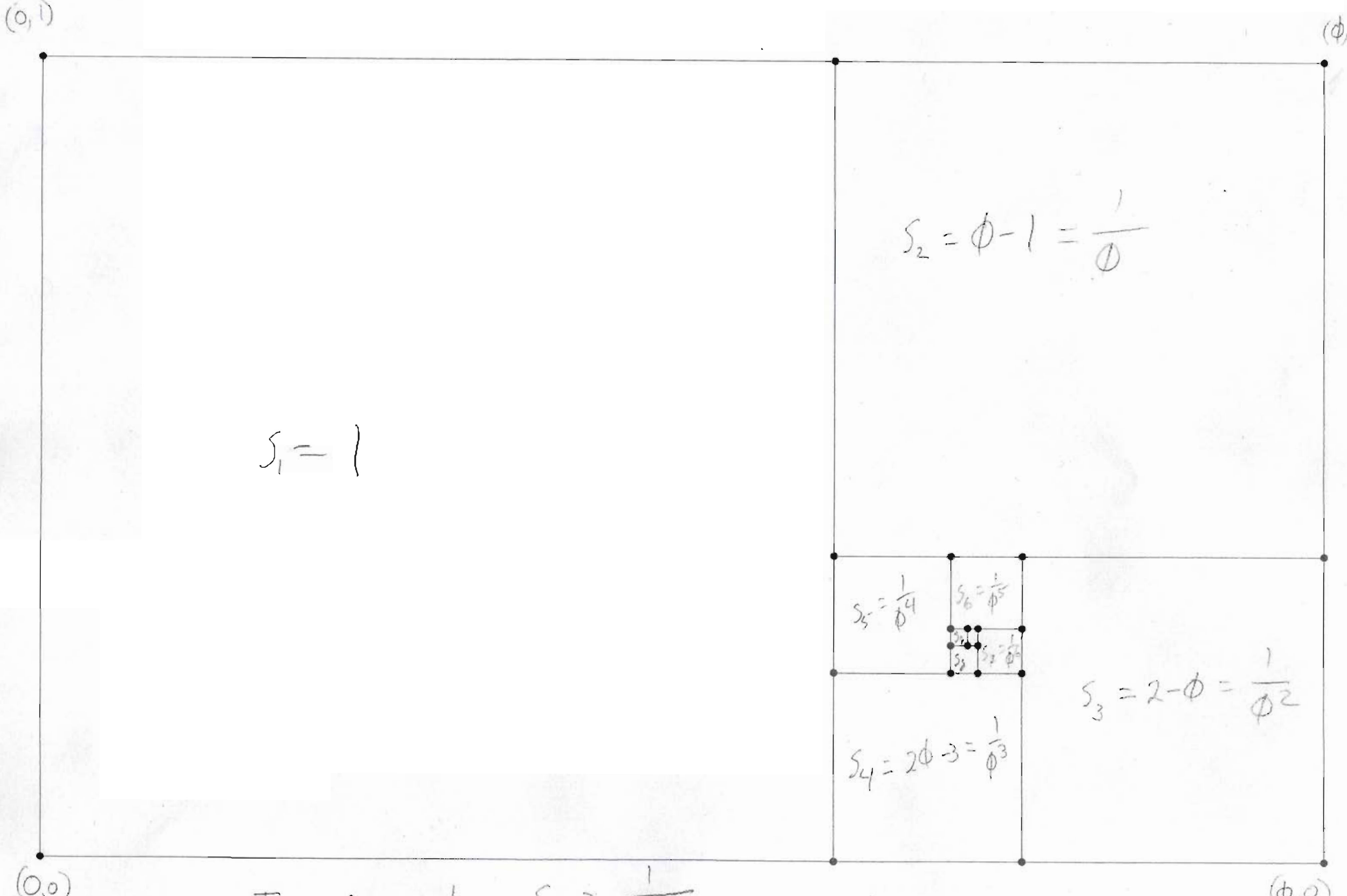
$\lim_{n\to\infty}(x_n) = \frac{\phi^2}{\sqrt{5}} \lim_{n\to\infty}\left(1 - \frac{1}{\phi^{4n}}\right)$ + $\lim_{n\to\infty}(y_n) = \frac{1}{\phi\sqrt{5}} \lim_{n\to\infty}\left(1 - \frac{1}{\phi^{4n}}\right)$

But $\lim_{n\to\infty}\left(1 - \frac{1}{\phi^{4n}}\right) = 0$. So $\lim_{n\to\infty}(x_n) = \frac{\phi^2}{\sqrt{5}} = \frac{1}{2} + \frac{3\sqrt{5}}{10}$

and $\lim_{n\to\infty}(y_n) = \frac{1}{\phi\sqrt{5}} = \frac{1}{2} - \frac{\sqrt{5}}{10}$. So, as before, the coordinates of the surviving point is $\left(\frac{1}{2} + \frac{3\sqrt{5}}{10}, \frac{1}{2} - \frac{\sqrt{5}}{10}\right)$

Note, this just increases the "ties that bind" between the Golden Ratio and the Fibonacci numbers.

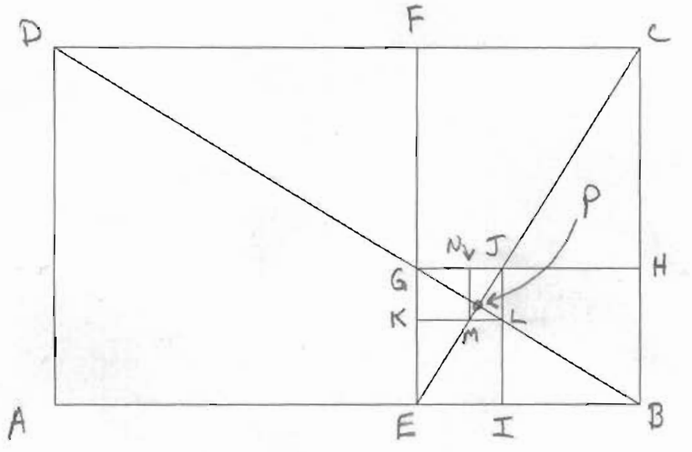
Tim "The Golden Boy" Noon



In general $S_n = \frac{1}{\phi^{n-1}}$

$\phi := \frac{1+\sqrt{5}}{2}$

Solution by Indira Dovegg



Solution: The point common to all rectangles in this sequence is the intersection P of the diagonals $\overline{BD}$ and $\overline{CE}$ of the first two rectangles in the sequence.

First note that all rectangles in this sequence are similar, that is they all have sides whose ratio is $\frac{1+\sqrt{5}}{2}$ to 1, the 'golden ratio'. The property that removing a square from a rectangle leaves a rectangle similar to the first is the property that defines the golden ratio.

Second note that the third rectangle in the sequence EBHG shares a corner with the first ABCD. Since they are similar the corner point G of the smaller one must lie on diagonal $\overline{BD}$ of the first. Likewise the fifth rectangle KLJG shares a corner with EBHG, so L lies on the diagonal $\overline{BG}$. In this way every odd-numbered rectangle in the sequence has two corners on $\overline{BD}$.

In exactly the same way the fourth rectangle EIJG shares a corner with the second EBCF, so J lies on $\overline{CE}$. The fifth rectangle MLJN shares a corner with EIJG, so M is on $\overline{EJ}$, etc, So every even-numbered rectangle has two corners on $\overline{CE}$.

Since the side lengths of the rectangles are all approaching 0 (the sequence of side lengths is geometric with $0<r<1$) the corners of the odd-numbered rectangles converge to a point on $\overline{BD}$ and the corners of the even ones converge to a point on $\overline{EC}$. The only point contained in all rectangles is $\overline{BD} \cap \overline{EC} = P$. If A is taken as the origin, the coordinates of P are $\left(\frac{5+3\sqrt{5}}{10}, \frac{5-\sqrt{5}}{10}\right) \approx (1.17, .276)$.